

Point processes in a metric space*

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Abstract

As a useful and elegant tool of extreme value theory, the study of point processes on a metric space is important and necessary for the analyses of heavy-tailed functional data. This paper focuses on the definition and properties of such point processes. A complete convergence result for a regularly varying iid sequence in a metric space is proved as an example of the application in extreme value theory.

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1 Introduction

Point process theory is considered as a useful tool in the analyses of extremal events of stochastic processes due to its close connection to the concept of regular variation, which plays an important role in modelling extremal events. For example, for an iid copies $(X_i)_{i \geq 1}$ of a random element X , it is well known that regular variation of X is equivalent to the convergence of point processes

$$N_n = \sum_{i=1}^n \delta_{(i/n, X_i/a_n)}$$

towards a suitable Poisson point process. This result is referred to as the complete convergence result; see for example [17]. A large number of literatures focus on the extension of this result and its applications; see [9, 10, 1, 3], to name a few. There are also a large amount of literatures on the applications of regular variation in extreme value theory, such as [12, 18, 2, 11], to name a few.

The classical formulation of regular variation for measures on a space such as \mathbb{R} and \mathbb{R}^d uses vague convergence; see for instance [17, 12] for more details. A generalization of vague convergence is $w^\#$ -convergence introduced by [5, 6] in which point process theory defined via $w^\#$ -convergence is well developed. Consider a complete and separable space (S, d) with a point $0_S \in S$. Let $M_b(S)$ denote the class of totally finite measures on $S = \mathcal{B}(S)$, let $M_+(S)$ denote the class of measures finite on compact Borel sets in S and let $M_\#(S)$ denote the class of measures finite on bounded Borel sets in S . One can define weak convergence, vague convergence, $w^\#$ -convergence in the corresponding class of measures $M_b(S)$, $M_+(S)$ and $M_\#(S)$. It should be noted as explained in [5] that

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$w^\#$ -convergence coincides with vague convergence when \mathbb{S} is locally compact. Borel sets bounded away from the origin are of interest in the field of extreme value theory and similar to the ideas of $w^\#$ -convergence, let $M_0 = M_0(\mathbb{S})$ denote the class of the measures finite on complements of neighborhoods of the point $0_{\mathbb{S}}$ and regular variation of measures in $M_0(\mathbb{S})$ is defined via the concept of M_0 -convergence; see [13] for details. In several aspects, M_0 -convergence has a lot of similarities to $w^\#$ -convergence and it allows us to use the results in [5, 6] to construct point process theory via M_0 -convergence.

The effort of generalizing the concept of M_0 -convergence focuses on allowing the introduction of hidden regular variation into the framework and studying simultaneously regular-variation properties at different scales. The basic idea is to removing a closed set C instead of one point $0_{\mathbb{S}}$, which leads to M_O -convergence where $O = \mathbb{S} \setminus C$; see [16] for a formal definition. Moreover, if C is a closed cone, M_O -convergence shares many properties with M_0 -convergence; see [13, 16]. For applications of the concept of M_O -convergence, it is necessary to develop point process theory as a useful and elegant tool. The basic idea of this paper is to define point processes via M_O -convergence and then to use them to analyse the heavy-tailed processes in a metric space.

The paper is arranged as follows. Section 2 gives an introduction to the spaces of measures on a complete and separable space, and M_O -convergence. Definitions and properties of random measures and point processes are given in Section 3. Poisson processes play an important role in applications of point processes, which is discussed in Section 4. The complete convergence result of point processes is proved under M_O -convergence in Section 5. Section 6 provides brief discussions on the choice of O and spaces of measures. Section 7 contains technical proofs of theorems in the previous sections.

2 The spaces of measures

Let (\mathbb{S}, d) be a complete and separable space and let $\mathcal{S} = \mathcal{B}(\mathbb{S})$ denote the Borel σ -field on \mathbb{S} , which is generated by open balls $B_r(x) = \{y \in \mathbb{S} : d(x, y) < r\}$ for $x \in \mathbb{S}$. Assume that $C \in \mathcal{S}$ is a closed subset and let $O = \mathbb{S} \setminus C$. The σ -algebra $\mathcal{S}_O = \{A \in \mathcal{S} : A \subset O\}$. Let $\mathcal{C}_O = \mathcal{C}_O(\mathbb{S})$ be a collection of real-valued, non-negative, bounded continuous functions f on O vanishing on $C^r = \{x \in \mathbb{S} : d(x, C) = \inf_{y \in C} d(x, y) < r\}$ for some $r > 0$. We say that a set $A \in \mathcal{S}_O$ is bounded away from C if $A \subset \mathbb{S} \setminus C^r$ for some $r > 0$. To define regular variation of measures in M_O , \mathbb{S} is assumed to be equipped with scalar multiplication; see Section 3.1, [16].

Definition 2.1. A scalar multiplication on \mathbb{S} is a map $[0, \infty) \times \mathbb{S} \rightarrow \mathbb{S} : (\lambda, x) \rightarrow \lambda x$ satisfying the following properties:

- (i) $\lambda_1(\lambda_2 x) = (\lambda_1 \lambda_2)x$ for all $\lambda_1, \lambda_2 \in [0, \infty)$ and $x \in \mathbb{S}$;
- (ii) $1x = x$ for $x \in \mathbb{S}$;
- (iii) the map is continuous with respect to the product topology;
- (iv) if $x \in O$ and if $0 \leq \lambda_1 < \lambda_2$, then $d(\lambda_1 x, C) < d(\lambda_2 x, C)$.

Assume additionally that C is a cone, that is, $\lambda C = C$ for $\lambda \in (0, \infty)$. Let $x \in O$. For any $\lambda \in [0, \infty)$, we have $\lambda(0x) = (\lambda 0)x = 0x$ by (i) in Definition 2.1. It follows that $d(\lambda_1(0x), C) = d(0x, C) = d(\lambda_2(0x), C)$ for all $\lambda_1, \lambda_2 \in (0, \infty)$. The condition (iv) in Definition 2.1 implies that $0x \in C$. If $x \in C$, $0x = 0(0x)$ and the above argument implies that $0x \in C$. So $0C \subset C$ and O is an open cone.

We always assume that the metric space (\mathbb{S}, d, C) is equipped with a scalar multiplication defined in Definition 2.1 and C is a closed cone. For simplicity, we will write the space (\mathbb{S}, d, C) as (\mathbb{S}, d) or \mathbb{S} .

2.1 The space \mathbb{M}_0 and \mathbb{M}_0 -convergence

Let $\mathbb{M}_0 = \mathbb{M}_0(\mathbb{S})$ be the space of Borel measures on \mathcal{S}_0 that are bounded on complements of \mathbb{C}^r , $r > 0$. The convergence $\mu_n \rightarrow \mu$ in \mathbb{M}_0 or $\mu_n \xrightarrow{M} \mu$ holds if and only if $\int f d\mu_n \rightarrow \int f d\mu$ for all $f \in \mathcal{C}_0$. Versions of the Portmanteau and continuous mapping theorem for \mathbb{M}_0 -convergence are stated as Theorem 2.1 and Theorem 2.3, respectively, in [16]. By choosing the metric

$$d_{\mathbb{M}_0}(\mu, \nu) = \int_0^\infty e^{-r} \frac{p_r(\mu^{(r)}, \nu^{(r)})}{1 + p_r(\mu^{(r)}, \nu^{(r)})}, \quad \mu, \nu \in \mathbb{M}_0,$$

where $\mu^{(r)}, \nu^{(r)}$ are the finite restriction of μ, ν to $\mathbb{S} \setminus \mathbb{C}^r$ and p_r is the Prohorov metric on the space of finite Borel measures on $\mathcal{B}(\mathbb{S} \setminus \mathbb{C}^r)$. It is shown in [16] that $d_{\mathbb{M}_0}(\mu, \nu) \rightarrow 0$ as $n \rightarrow \infty$ if and only if $\mu_n \rightarrow \mu$ in \mathbb{M}_0 and the space $(\mathbb{M}_0, d_{\mathbb{M}_0})$ is complete and separable. Denote \mathcal{M}_0 as the σ -algebra of \mathbb{M}_0 , which is generated by the neighborhoods of $\mu \in \mathbb{M}_0$

$$\left\{ \nu \in \mathbb{M}_0 : \left| \int f_i d\nu - \int f_i d\mu \right| < \varepsilon, \quad i = 1, \dots, k \right\},$$

where $\varepsilon > 0$ and $f_i \in \mathcal{C}_0$ for $i = 1, \dots, k$.

Proposition 2.2. *The Borel σ -algebra \mathcal{M}_0 is the smallest σ -algebra with respect to which the mappings $\Phi_A : \mathbb{M}_0 \mapsto \mathbb{R} \cup \{\pm\infty\}$ given by*

$$\Phi_A(\mu) = \mu(A),$$

are measurable for all sets A in a π -system \mathcal{D} generating \mathcal{S}_0 and in particular for the sets $A \in \mathcal{S}_0$ bounded away from \mathbb{C} .

The proof is technical and is put in Section 7.

2.2 The space \mathbb{N}_0

Let \mathbb{N}_0 be the space of all measures $N \in \mathbb{M}_0$ satisfying that for each $r > 0$, $N(A \setminus \mathbb{C}^r)$ is a non-negative integer for all $A \in \mathcal{S}$. We call the measure N a counting measure for short. For $x \in \mathcal{S}_0$ and $\mu \in \mathbb{M}_0$, we say that the measure μ has an atom x if $\mu(\{x\}) > 0$. A measure with only atoms is *purely atomic*, while a *diffuse* measure has no atom. We use δ_x to denote Dirac measure at $x \in \mathbb{O}$, defined on $A \in \mathcal{S}_0$ by

$$\delta_x(A) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

The following lemma shows that the counting measure is purely atomic.

Lemma 2.3. *Assume that the measure $\mu \in \mathbb{M}_0$.*

(i) *The measure μ is uniquely decomposable as $\mu = \mu_a + \mu_d$, where*

$$\mu_a = \sum_{i=1}^{\infty} \kappa_i \delta_{x_i} \tag{2.1}$$

is a purely atomic measure, uniquely determined by a countable set $\{(x_i, \kappa_i)\} \subset \mathbb{O} \times (0, \infty)$, and μ_d is a diffuse measure.

(ii) *A measure $N \in \mathbb{M}_0$ is a counting measure if and only if (1) its diffuse component is null, (2) all κ_i in (2.1) are positive integers, and (3) the set $\{x_i\}$ defined in (2.1) is a countable set with at most finite many x_i in any set $\mathbb{S} \setminus \mathbb{C}^r$ with $r > 0$.*

Proof. Let $r_j = 1/j$, $j = 1, 2, \dots$. Let $\mathcal{O}^{(1)} = \mathbb{S} \setminus \mathbb{C}^{r_1}$ and $\mathcal{O}^{(j+1)} = \mathbb{C}^{r_j} \setminus \mathbb{C}^{r_{j+1}}$, $j = 1, 2, \dots$. Then, $\mathcal{O} = \bigcup_{j=1}^{\infty} \mathcal{O}^{(j)}$. By definition of $\mathbb{M}_{\mathcal{O}}$, if $\mu \in \mathbb{M}_{\mathcal{O}}$, the measure $\mu_j(\cdot) = \mu(\cdot \cap \mathcal{O}^{(r_j)})$ and hence μ is σ -finite. Part (i) is a property of σ -finite measures; see Appendix A1.6, [5] for details.

Since μ_j is finite, Proposition 9.1.III in [6] implies that μ_j is a counting measure if and only if all the three conditions in (ii) are satisfied. Moreover, if μ_j is a counting measure, all of its atoms must lie in $\mathcal{O}^{(r_j)}$. Because $\mathcal{O}^{(r_j)}$ are disjoint sets and $\mu = \sum_{j=1}^{\infty} \mu_j$, the measure μ is a counting measure if and only if all the three conditions in (ii) are satisfied. \square

The following theorem is an application of Lemma 2.3.

Theorem 2.4. $\mathbb{N}_{\mathcal{O}}$ is a closed subset of $\mathbb{M}_{\mathcal{O}}$.

Proof. It is enough to show that the limit of a sequence in $\mathbb{N}_{\mathcal{O}}$ is still in $\mathbb{N}_{\mathcal{O}}$. Let $(N_k)_{k \in \mathbb{N}}$ be a sequence of counting measures and $N_k \rightarrow N$ in $\mathbb{M}_{\mathcal{O}}$. Let y be an arbitrary point in \mathcal{O} . Since $N \in \mathbb{M}_{\mathcal{O}}$, for all but a countable set of values of $r \in (0, d(y, \mathbb{C}))$, $N(\partial B_r(y)) = 0$, where ∂A is the boundary of a set $A \in \mathcal{S}_{\mathcal{O}}$. We can find a decreasing sequence $(r_j)_{j \in \mathbb{N}}$ such that $\lim_{j \rightarrow \infty} r_j = 0$ and $N(\partial B_{r_j}(y)) = 0$, $j \geq 1$. By Portmanteau theorem (Theorem 2.1, [16]), we have that for $j \geq 1$,

$$N_k(B_{r_j}(y)) \rightarrow N(B_{r_j}(y)), \quad k \rightarrow \infty.$$

Since $N_k(B_{r_j}(y))$ are non-negative integers, $N(B_{r_j}(y))$ are also non-negative integers and thus N is a counting measure by Lemma 2.3. \square

3 Random measures, point processes and weak convergence

In this section, the properties of random measures and point processes are studied. We will show that weak convergence is determined by the finite-dimensional convergence. By applying this result, it can be shown that weak convergence is equivalent to convergence of Laplace functionals, which will be frequently used in the following sections.

Definition 3.1. (i) A random measure ξ with state space \mathcal{O} is a measurable mapping from a probability space (Ω, \mathcal{E}, P) into $(\mathbb{M}_{\mathcal{O}}, \mathcal{M}_{\mathcal{O}})$.

(ii) A point process on \mathcal{O} is a measurable mapping from a probability space (Ω, \mathcal{A}, P) into $(\mathbb{N}_{\mathcal{O}}, \mathcal{N}_{\mathcal{O}})$.

A realization of a random measure ξ has the value $\xi(A, \omega)$ on the borel set $A \in \mathcal{S}_{\mathcal{O}}$. For each fixed A , $\xi_A = \xi(A, \cdot)$ is a function mapping Ω into $\mathbb{R}^+ = [0, \infty]$. The following proposition provides a convenient way to examine whether a mapping is a random measure.

Theorem 3.2. Let ξ be a mapping from a probability space (Ω, \mathcal{E}, P) into $\mathbb{M}_{\mathcal{O}}$ and \mathcal{D} be a π -system of Borel sets bounded away from \mathbb{C} , which generates $\mathcal{S}_{\mathcal{O}}$. Then ξ is a random measure if and only if ξ_A is a random variable for each $A \in \mathcal{D}$. Similarly, N is a point process if and only if $N(A)$ is a random variable for each $A \in \mathcal{D}$.

Proof. Let \mathcal{U} be the σ -algebra of subsets of $\mathbb{M}_{\mathcal{O}}$ whose inverse images under ξ are events, and let Φ_A denote the mapping taking a measure $\mu \in \mathbb{M}_{\mathcal{O}}$ into $\mu(A)$. Because $\xi_A(\omega) = \xi(A, \omega) = \Phi_A(\xi(\cdot, \omega))$,

$$\xi^{-1}(\Phi_A^{-1}(B)) = (\xi_A)^{-1}(B), \quad B \in \mathcal{B}(\mathbb{R}^+).$$

If ξ_A is a random variable, $(\xi_A)^{-1}(B) \in \mathcal{E}$ and we have $\Phi_A^{-1}(B) \in \mathcal{U}$ by definition. Proposition 2.2 yields that $\mathcal{M}_{\mathcal{O}} \subset \mathcal{U}$ and thus ξ is a random measure. Conversely, if ξ

is a random measure, $\Phi_A^{-1}(B) \in \mathcal{M}_0$ for $B \in \mathcal{B}(\mathbb{R}^+)$ and hence, $\xi^{-1}(\Phi_A^{-1}(B)) \in \mathcal{E}$. This shows that ξ_A is a random variable.

It is similar to show that N is a point process if and only if $N(A)$ is a random variable for each $A \in \mathcal{D}$. \square

As a simple application of Proposition 2.2 and Theorem 3.2, it is easy to prove the following corollary and hence the proof is omitted.

Corollary 3.3. *The sufficient and necessary condition for ξ to be a random measure is that $\xi(A)$ is a random variable for each $A \in \mathcal{S}_0$ bounded away from \mathbb{C} . Similarly, the sufficient and necessary condition for N to be a point process is that $N(A)$ is a random variable for each $A \in \mathcal{S}_0$ bounded away from \mathbb{C} .*

3.1 Finite-dimensional distributions

The finite-dimensional, or *fidi* for short, distributions of a random measure ξ are the joint distributions for all finite families of the random variables $\xi(A_1), \dots, \xi(A_k)$, where A_1, \dots, A_k are Borel sets bounded away from \mathbb{C} , that is, the family of proper distribution functions

$$F_k(A_1, \dots, A_k; x_1, \dots, x_k) = P(\xi(A_i) \leq x_i, i = 1, \dots, k). \quad (3.1)$$

Theorem 3.4. *The distribution of a random measure is completely determined by the fidi distributions (3.1) for all finite families $\{A_1, \dots, A_k\}$ of disjoint sets from a π -system of Borel sets bounded away from \mathbb{C} generating \mathcal{S}_0 .*

Note that for a π -system \mathcal{A} , if two probability measures P_1 and P_2 agree on \mathcal{A} , then P_1 and P_2 agree on $\sigma(\mathcal{A})$; see Theorem 3.3, [4]. The Borel sets bounded away from \mathbb{C} forms a π -system and hence Theorem 3.4 holds by Proposition 2.2. The proof is completed.

Similarly, the fidi distributions of the point process N are the joint distributions for Borel sets bounded away from \mathbb{C} , $\{A_1, \dots, A_k\}$ and nonnegative integers n_1, n_2, \dots , is defined by

$$P_k(A_1, \dots, A_k; n_1, \dots, n_k) = P(N(A_i) = n_i, i = 1, \dots, k).$$

According to Theorem 3.4, the fidi distributions determine N .

3.2 Laplace functionals

Let $BM(\mathbb{S})$ be a class of non-negative bounded measurable functions f for which there exists $r > 0$ such that f vanishes on \mathbb{C}^r . Let $\xi : (\Omega, \mathcal{E}, P) \rightarrow (\mathbb{M}_0, \mathcal{M}_0)$ be a random measure and N be a point process. The Laplace functional is defined for a random measure ξ and each $f \in BM(\mathbb{S})$ by

$$L_\xi[f] = E[\exp(-\xi(f))] = \int_{\mathbb{M}_0} \exp\left(-\int_{\mathbb{O}} f(x) \xi(dx)\right) P(d\xi), \quad (3.2)$$

where $\xi_f = \int_{\mathbb{O}} f(x) \xi(dx)$. Similarly, we can define $L_N[f]$ for the point process N and each $f \in BM(\mathbb{S})$.

Theorem 3.5. *The Laplace functions $\{L_\xi[f] : f \in BM(\mathbb{S})\}$ uniquely determine the distribution of a random measure ξ . Similarly, the Laplace functions $\{L_N[f] : f \in BM(\mathbb{S})\}$ uniquely determine the distribution of a point process N .*

Proof. For $k \geq 1$ and Borel sets $A_1, \dots, A_k \in \mathcal{S}_0$ bounded away from \mathbb{C} and $\lambda_i > 0$, $i = 1, \dots, k$, the function $f : \mathbb{O} \rightarrow [0, \infty)$ is given by

$$f(x) = \sum_{i=1}^k \lambda_i \mathbf{1}_{A_i}(x), \quad x \in \mathbb{O},$$

where $\mathbf{1}_A$ is the indicator function for $A \in \mathcal{S}_0$. Then for each realization $\omega \in \Omega$,

$$\xi(\omega, f) = \int_{\mathbb{S}} f(x) \xi(\omega, dx) = \sum_{i=1}^k \lambda_i \xi(\omega, A_i),$$

and

$$L_\xi[f] = E \exp \left(- \sum_{i=1}^k \lambda_i \xi(A_i) \right),$$

which is the joint Laplace transform of the random vector $(\xi(A_i))_{i=1, \dots, k}$. The uniqueness of theorem for Laplace transform for random vectors yields that L_ξ uniquely determines the law of $(\xi(A_i))_{i=1, \dots, k}$ and Theorem 3.4 completes the proof. \square

The following proposition shows the convergence of Laplace functionals.

Proposition 3.6. *Let ξ be a random measure. For a sequence of functions $(f_n)_{n \in \mathbb{N}}$ with $f_n \in BM(\mathbb{S})$ and a function $f \in BM(\mathbb{S})$, the convergence $L_\xi[f_n] \rightarrow L_\xi[f]$ as $\sup_{x \in \mathbb{O}} |f_n(x) - f(x)| \rightarrow 0$ if one of three conditions holds: (i) $\xi(\mathbb{O}) < \infty$; (ii) the pointwise convergence $f_n \rightarrow f$ is monotonic; (iii) there exists $r > 0$ such that for each $n \geq 1$, f_n vanishes on \mathbb{C}^r .*

Proof. If condition (i) holds,

$$|L_\xi[f_n] - L_\xi[f]| \leq \xi(\mathbb{O}) \sup_{x \in \mathbb{O}} |f_n(x) - f(x)| \rightarrow 0.$$

If condition (iii) holds, since $\xi(\mathbb{S} \setminus \mathbb{C}^r) < \infty$,

$$|L_\xi[f_n] - L_\xi[f]| \leq \xi(\mathbb{S} \setminus \mathbb{C}^r) \sup_{x \in \mathbb{O}} |f_n(x) - f(x)| \rightarrow 0.$$

Suppose that condition (ii) holds. If $f_n(x) \downarrow f(x)$ for each $x \in \mathbb{O}$, this implies that there exists $r > 0$ that f_n vanishes on \mathbb{C}^r because $f_1 \in BM(\mathbb{S})$ and condition (iii) is satisfied. If $f_n(x) \uparrow f(x)$ for each $x \in \mathbb{O}$, dominated convergence theorem ensures that $L_\xi[f_n] \rightarrow L_\xi[f]$. \square

3.3 Weak convergence of random measures

Weak convergence is characterized by weak convergence of fidi distributions. In connection of this idea, we are interested in the *stochastic continuity sets*, that is, Borel sets $A \in \mathcal{S}_0$ bounded away from \mathbb{C} satisfying the condition that $P(\xi(\partial A) > 0) = 0$ for a probability measure P . Let S_P be the collection of such sets. It is trivial to show that for $A, B \in S_P$, $A \cup B \in S_P$ and $A \cap B \in S_P$. Hence, S_P is a π -system. The following lemma implies that S_P generates \mathcal{S}_0 .

Lemma 3.7. *Let P be a probability measure on \mathcal{M}_0 and S_P be the class of stochastic continuity sets for P . Then for all but a countable set of values of $r > 0$, $\mathbb{C}^r \in S_P$ and given $x \in \mathbb{O}$, for all but a countable set of values of $r \in (0, d(x, \mathbb{C}))$, $B_r(x) \in S_P$.*

Proof. It is sufficient to show that for each finite positive $\varepsilon_1, \varepsilon_2$ and r_0 , there are only finite numbers of $r > r_0$ satisfying

$$P\left(\xi\left(\partial(\mathbb{C}^r)\right) > \varepsilon_1\right) > \varepsilon_2. \quad (3.3)$$

Suppose the contrary and there exists positive numbers $\varepsilon_1, \varepsilon_2$ and r_0 such that there is a countably infinite set $\{r_i, i \geq 1\}$ with $r_i > r_0$ satisfying

$$P(A_i) > \varepsilon_2, \quad i \geq 1,$$

where $A_i = \{\xi \in \mathcal{M}_0 : \xi(\partial \mathbb{C}^{r_i}) > \varepsilon_1\}$. Therefore,

$$\varepsilon_2 \leq \limsup_{i \rightarrow \infty} P(A_i) \leq P(\limsup_{i \rightarrow \infty} A_i) \leq P(\xi(\mathbb{S} \setminus \mathbb{C}^{r_0}) = \infty).$$

This contradicts to the assumption that for any $\xi \in \mathcal{M}_0$ and $r > 0$, $\xi(\mathbb{S} \setminus \mathbb{C}^r) < \infty$. Thus, (3.3) holds for a finite number of $r > r_0$.

Applying a similar arguments, we can easily show that given $x \in \mathcal{O}$, for all but a countable set of values of $r \in (0, d(x, \mathbb{C}))$, $B_r(x) \in S_P$. \square

We say that a sequence of random measures $(\xi_n)_{n \geq 1}$ converges in the sense of fidi distributions if for every finite family $\{A_1, \dots, A_k\}$ with $A_i \in S_P$, the joint distributions of $(\xi_n(A_1), \dots, \xi_n(A_k))$ converge weakly in $\mathcal{B}(\mathbb{R}^k)$ to the joint distribution of $(\xi(A_1), \dots, \xi(A_k))$. Let $(P_n)_{n \geq 1}$ be a sequence of probability measures and assume that $P_n \rightarrow P$ weakly, or $P_n \xrightarrow{w} P$. Let A be a stochastic continuity set for P and denote the mapping $\Phi_A : \xi \mapsto \xi(A)$. According the definition of the stochastic continuity set, $P(D) = 0$ where D is the set of discontinuity points for Φ_A . By continuous mapping theorem (see Proposition A2.3.V in [5]),

$$P_n(\Phi_A^{-1}) \rightarrow P(\Phi_A^{-1}).$$

Similarly, for any finite family $\{A_1, \dots, A_k\}$ with $A_i \in S_P$, let $\Phi_k : \xi \mapsto (\xi(A_1), \dots, \xi(A_k))$ and $P_n(\Phi_k^{-1}) \rightarrow P(\Phi_k^{-1})$ if $P_n \xrightarrow{w} P$. This leads to the following lemma.

Lemma 3.8. *Weak convergence implies weak convergence of the finite-dimensional convergence.*

To show that weak convergence of the finite-dimensional convergence implies weak convergence, we need to show that the converging sequence of probability measures are *uniformly tight*. A family of probability measures $(P_t)_{t \in \mathcal{T}}$ with an index set \mathcal{T} on \mathcal{M}_0 is *uniformly tight* if for each $\varepsilon > 0$, there exists a compact set $K \in \mathcal{M}_0$ such that $P_t(K) > 1 - \varepsilon$ for all $t \in \mathcal{T}$.

Lemma 3.9. *For a family of probability measures $(P_t)_{t \in \mathcal{T}}$ on \mathcal{M}_0 to uniformly tight, it is necessary and sufficient that there exists a sequence (r_i) with $r_i \downarrow 0$ such that for each i and any $\varepsilon, \varepsilon' > 0$, there exist real numbers $M_i > 0$ and compact set $K_i \subset \mathbb{S} \setminus \mathbb{C}^{r_i}$ such that, uniformly for $t \in \mathcal{T}$,*

$$P_t(\xi(\mathbb{S} \setminus \mathbb{C}^{r_i}) > M_i) < \varepsilon, \quad (3.4)$$

$$P_t(\xi(\mathbb{S} \setminus (\mathbb{C}^{r_i} \cup K_i)) < \varepsilon') < \varepsilon. \quad (3.5)$$

The proof of Lemma 3.9 is in Section 7. The next theorem is the main result in this section and its proof is in Section 7.

Theorem 3.10. *Let $(P_n)_{n \geq 1}$ and P be distributions on \mathcal{M}_0 . Then, $P_n \xrightarrow{w} P$ if and only if the fidi distribution of P_n converge weakly to those of P .*

Two equivalent conditions for weak convergence can be derived as the corollaries to Theorem 3.10.

Corollary 3.11. *The two following conditions is equivalent to the weak convergence $P_n \rightarrow P$ with $f \in \mathcal{C}_0$:*

- (i) *The distribution of $\int_0 f d\xi$ under P_n converges weakly to its distribution under P .*
- (ii) *The Laplace functionals $L_n[f] = E_{P_n} \left(\exp(-\int_0 f(x)\xi(dx)) \right)$ converge pointwise to the limit functional $L[f] = E_P \left(\exp(-\int_0 f(x)\xi(dx)) \right)$.*

Proof. Condition (ii) is equivalent to (i) by well-known results on Laplace transform.

Think of the simple functions of the form $f = \sum_{i=1}^k c_i \mathbf{1}_{A_i}$, where k is a finite positive integer, $\sum_i |c_i| < \infty$ and $(A_i)_{i \geq 1}$ are a family of Borel sets with $A_i \in \mathcal{S}_P$. Convergence of distributions of the integrals $\int_{\mathcal{O}} f d\xi$ is equivalent to the finite-dimensional convergence for every finite k . Following a classical arguments, we can find $h_l^+, h_l^- \in \mathcal{C}_0$ satisfying that $0 < h_l^-(x) \uparrow f(x)$ and $h_l^+(x) \downarrow f(x)$ holds uniformly for every $x \in \mathcal{S}$ as $l \rightarrow \infty$. It follows that condition (i) and Proposition 3.6 implies the finite-dimensional convergence and hence weak convergence. \square

4 Poisson processes

Poisson processes play a vital role in the applications of the point process and we start with the definition of a Poisson process.

Definition 4.1. Given a random measure $\mu \in \mathbb{M}_0$, a point process N is called a Poisson process or Poisson random measure (PRM) with mean measure μ if N satisfies

(i) For any $A \in \mathcal{S}_0$ and any non-negative integer k ,

$$P(N(A) = k) = \begin{cases} \exp(\mu(A))(\mu(A))^k / k! , & \mu(A) < \infty , \\ 0 , & \mu(A) = \infty . \end{cases}$$

(ii) For any $k \geq 1$, if A_1, \dots, A_k are mutually disjoint Borel sets bounded away from \mathbb{C} , then $N(A_i)$, $i = 1, \dots, k$ are independent random variables.

For short, we write Poisson process with mean measure μ as **PRM**(μ).

Proposition 4.2. PRM(μ) exists and its law is uniquely determined by conditions (i) and (ii) in Definition 4.1. Moreover, the Laplace functional of PRM(μ) is given by

$$L_N[f] = \exp \left(- \int_{\mathcal{O}} (1 - e^{-f(x)}) \mu(dx) \right), \quad f \in BM(\mathcal{S}), \quad (4.1)$$

and conversely a point process with Laplace functional of the form (4.1) must be PRM(μ).

Sketch of the proof. To prove that the Laplace functional of PRM(μ) is given by (4.1), one can choose $f = c \mathbf{1}_A$ for $c \geq 0$ and $A \in \mathcal{S}_0$ bounded away from \mathbb{C} . Following the lines in the proof of Proposition 3.6, [17], the Laplace functional $L_N[f]$ has the form (4.1). Let

$$f = \sum_{i=1}^k c_i \mathbf{1}_{A_i}, \quad (4.2)$$

where $k > 0$, $c_i \geq 0$ and A_1, \dots, A_k are disjoint sets bounded away from \mathbb{C} . Similarly, it can be shown that the Laplace functional $L_N[f]$ has the form (4.1). Then for any $f \in BM(\mathcal{S})$, there exists simple functions f_n of the form (4.2) such that $f_n \uparrow f$ with $\sup_{x \in \mathcal{O}} |f_n(x) - f(x)| \rightarrow 0$ as $n \rightarrow \infty$. By Proposition 3.6, we have that the Laplace functional $L_N[f]$ has the form (4.1). Conversely, it is easy to prove following the lines of the lines in the proof of Proposition 3.6, [17].

The proof of the existence of PRM(μ) is through construction. we will use the same trick in the proof of Lemma 2.3 to divide \mathcal{O} into countable disjoint subspaces $\mathcal{O}^{(r_j)}$, $j = 1, 2, \dots$. Then let $\mu_j(\cdot) = \mu(\cdot \cap \mathcal{O}^{(r_j)})$ for $\mu \in \mathbb{M}_0$. Using the arguments in the proof of Proposition 3.6, [17], it is easy to construct PRM(μ_i) for $i \geq 1$, named N_i . Let

$N = \sum_{i=1}^{\infty} N_i$. For $f \in BM(\mathbb{S})$,

$$\begin{aligned} L_N[f] &= E \exp\left(-\sum_{i=1}^{\infty} N_i(f)\right) = \lim_{n \rightarrow \infty} E \exp\left(-\sum_{i=1}^n N_i(f)\right) \\ &= \lim_{n \rightarrow \infty} \prod_{i=1}^n E \exp(-N_i(f)) \\ &= \lim_{n \rightarrow \infty} \prod_{i=1}^n E \exp\left(-\int_{\mathbb{O}} (1 - e^{-f(x)}) \mu_i(dx)\right) \\ &= \exp\left(-\int_{\mathbb{O}} (1 - e^{-f(x)}) \sum_{i=1}^{\infty} \mu_i(dx)\right) \\ &= \exp\left(-\int_{\mathbb{O}} (1 - e^{-f(x)}) \mu(dx)\right). \end{aligned}$$

This shows that $\text{PRM}(\mu)$ exists. \square

A new Poisson process can be constructed by mapping points of a Poisson process. Recall that the Dirac measure δ_x for $x \in \mathbb{S}$ is defined by

$$\delta_x(A) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A, \end{cases} \quad A \in \mathcal{S}.$$

Proposition 4.3. Assume that the complete and separable spaces (\mathbb{S}_1, d_1) and (\mathbb{S}_2, d_2) with closed cones $\mathbb{C}_1 \subset \mathbb{S}_1$ and $\mathbb{C}_2 \subset \mathbb{S}_2$ are equipped with scalar multiplication. Let $\mathbb{O}_1 = \mathbb{S}_1 \setminus \mathbb{C}_1$ and $\mathbb{O}_2 = \mathbb{S}_2 \setminus \mathbb{C}_2$. Denote $\mathcal{S}_{\mathbb{O}_i}$, $i = 1, 2$ as the corresponding σ -algebra of \mathbb{O}_i , $i = 1, 2$, correspondingly. Let $T : (\mathbb{S}_1, \mathcal{S}_{\mathbb{O}_1}) \rightarrow (\mathbb{S}_2, \mathcal{S}_{\mathbb{O}_2})$ be a measurable mapping satisfying the condition that for every $\varepsilon > 0$,

$$\inf\{d_1(x, \mathbb{C}_1) : d_2(Tx, \mathbb{C}_2) > \varepsilon\} > 0. \quad (4.3)$$

Then if N is $\text{PRM}(\mu)$ on \mathbb{O}_1 , then

$$\tilde{N}_1 = N \circ T^{-1}$$

is $\text{PRM}(\tilde{\mu})$ on \mathbb{O}_2 with $\tilde{\mu} = \mu \circ T^{-1}$. If we have a representation

$$N = \sum_i \delta_{X_i},$$

then $\tilde{N} = N \circ T^{-1} = \sum_i \delta_{TX_i}$.

Proof. Let $f \in BM(\mathbb{S}_2)$ and there exist $r > 0$ such that f vanishes on $\mathbb{C}_2^r = \{x \in \mathbb{O}_2 : d(x, \mathbb{C}_2) < r\}$. The condition (4.3) implies that for $r > 0$, there exists $\tilde{r} > 0$ such that $T^{-1}(x_2) \in \mathbb{S}_1 \setminus \mathbb{C}_1^{\tilde{r}}$ for $x \in \mathbb{S}_2 \setminus \mathbb{C}_2^r$. Therefore, $f \circ T \in BM(\mathbb{S}_1)$ and $\mu \circ T^{-1} \in \mathcal{M}_{\mathbb{O}_2}$. An application of Proposition 4.2 yields that

$$\begin{aligned} L_{\tilde{N}}[f] &= E \exp(-\tilde{N}(f)) = E \exp\left(-\int_{\mathbb{O}_2} f(x_2) N \circ T^{-1}(\omega, dx_2)\right) \\ &= E \exp\left(-\int_{\mathbb{O}_1} f(Tx_1) N(\omega, dx_1)\right) \\ &= \exp\left(-\int_{\mathbb{O}_1} (1 - e^{-f \circ T}) d\mu\right) \\ &= \exp\left(-\int_{\mathbb{O}_2} (1 - e^{-f(x)}) \mu \circ T^{-1}(dx)\right), \end{aligned}$$

which is the Laplace functional of $\text{PRM}(\mu \circ T^{-1})$ on \mathbb{S}_2 . \square

We next construct a new PRM living in a higher dimensional space from a given PRM. Let (\mathbb{S}, d_1) with a closed cone \mathbb{C} be a complete and separable space and let (\mathbb{K}, d_2) be a complete and separable space. Due to the separability of \mathbb{K} , there exists a countable dense $D \subset \mathbb{K}$ and $r_0 > 0$ such that the open neighborhoods $\{y \in \mathbb{K} : d(x, y) < r_0\}$ for each $x \in D$ covers \mathbb{K} . Denote \mathcal{K} as the Borel σ -algebra on \mathbb{K} . We define a transition function $G : \mathbb{K} \times \mathcal{O} \rightarrow [0, 1]$ from \mathcal{O} to \mathbb{K} : $G(F, \cdot)$ is \mathcal{S} -measurable for $F \in \mathcal{K}$ and $G(\cdot, x)$ is a probability measure on \mathcal{K} for each $x \in \mathcal{O}$. Assume that the sequence of random elements $(K_n)_{n \geq 1}$ taking values in \mathbb{K} is conditionally independent given the sequence $(X_n)_{n \geq 1}$ taking values in \mathcal{O} , that is,

$$P(K_i \in F \mid (X_n)_{n \geq 1}, (K_j)_{j \neq i}) = G(X_i, F), \quad i \geq 1, F \in \mathcal{K}. \quad (4.4)$$

Consider the space $\tilde{\mathbb{S}} = (\mathbb{K}, \mathbb{S})$ assigned with the Euclidean distance

$$\tilde{d}((x_1, y_1), (x_2, y_2)) = \sqrt{(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2}, \quad (4.5)$$

where $(x_i, y_i) \in \mathbb{K} \times \mathbb{S}$, $i = 1, 2$. Let $\tilde{\mathbb{C}} = \mathbb{K} \times \mathbb{C}$ and $\tilde{\mathcal{O}} = \tilde{\mathbb{S}} \setminus \tilde{\mathbb{C}}$. The set $\tilde{\mathbb{C}}$ is closed. Now we define a scalar multiplication on $\tilde{\mathbb{S}}$ as a map $[0, \infty) \times \tilde{\mathbb{S}} \rightarrow \tilde{\mathbb{S}} : (\lambda, (x, y)) \rightarrow (\lambda x, y)$. It is trivial to show that the conditions (i), (iii) and (iii) in Definition 2.1 are satisfied. Since $\tilde{d}((x, y), \tilde{\mathbb{C}}) = d_1(x, \mathbb{C})$ for $(x, y) \in \tilde{\mathbb{S}}$, the condition (iv) in Definition 2.1 is also satisfied. Moreover, $\tilde{\mathbb{C}}$ is also a cone. According to the arguments above, the space $(\mathbb{M}_{\tilde{\mathcal{O}}}, \mathcal{M}_{\tilde{\mathcal{O}}})$ is also well defined. The set

$$\tilde{\mathbb{C}}^r = \{(x, y) : \tilde{d}((x, y), \tilde{\mathbb{C}}) < r\} = \{(x, y) : d_1(x, \mathbb{C}) < r\},$$

is then an open set. Let $BM(\tilde{\mathbb{S}})$ be a class of the bounded and measurable functions $f : \tilde{\mathbb{S}} \rightarrow [0, \infty)$ vanishing on $\tilde{\mathbb{C}}^r$ for some $r > 0$. Denote $\tilde{\mathcal{S}}_{\tilde{\mathcal{O}}}$ as the Borel σ -algebra of $\tilde{\mathcal{O}}$ generated by a π -system consisting of the sets of the form $A \times F$ with $F \in \mathcal{K}$ and $A \in \mathcal{B}(\mathbb{C}^r)$ for $r > 0$.

Similar to Lemma 3.9 and 3.10 in [17], we have the following lemma.

Lemma 4.4. Assume that $f : \tilde{\mathcal{O}} \rightarrow [0, 1]$ belongs to $BM(\tilde{\mathbb{S}})$. Then for $i \geq 1$,

$$E(f(K_i, X_i) \mid (X_n)_{n \geq 1}) = \int_{\mathbb{K}} f(y, X_i) G(dy, X_i), \quad a.s., \quad (4.6)$$

and

$$E\left(\prod_{i=1}^{\infty} f(K_i, X_i) \mid (X_n)_{n \geq 1}\right) = \prod_{i=1}^{\infty} E(f(K_i, X_i) \mid (X_n)_{n \geq 1}), \quad a.s.. \quad (4.7)$$

The following proposition is proved by using Laplace functional and the proof is similar to the proof of Proposition 3.8 in [17]. Therefore, the proof is omitted.

Proposition 4.5. Let $\tilde{\mathbb{S}} = (\mathbb{S}, \mathbb{J})$ with K as the transition function be the space defined before Lemma 4.4. Suppose that

$$N = \sum_i \delta_{X_i}$$

is PRM(μ) on (\mathbb{S}, d_1) . Then

$$N^* = \sum_i \delta_{(X_i, K_i)}$$

is PRM on $\tilde{\mathbb{S}}$ with mean measure

$$\mu^*(dx, dy) = \mu(dx)G(x, dy).$$

5 Regular variation

For $\tau \in \mathbb{R}$, let \mathcal{R}_τ denote the class of regularly varying functions at infinity with index τ , i.e., positive, measurable functions g defined in a neighbourhood of infinity such that $\lim_{u \rightarrow \infty} g(\lambda u)/g(u) = \lambda^\tau$ for every $\lambda \in (0, \infty)$.

Definition 5.1. A sequence of measures $(\nu_n)_{n \geq 1}$ in \mathbb{M}_0 is regularly varying if there exists a nonzero $\mu \in \mathbb{M}_0$ and a regularly varying function b with index $-\alpha^{-1}$, $\alpha > 0$ such that $t\nu(b(t) \cdot) \rightarrow \mu$ in \mathbb{M}_0 as $t \rightarrow \infty$.

The limiting measure μ in Definition 5.1 has the homogeneity property

$$\mu(\lambda A) = \lambda^{-\alpha} \mu(A), \quad \alpha > 0, \lambda > 0, A \in \mathcal{S}_0. \quad (5.1)$$

This property is proved by Theorem 3.1 in [16]. Similarly, a random element X in \mathbb{S} is regularly varying with index $-\alpha$ if and only if there exists a nonzero $\mu \in \mathbb{M}_0$ and a regularly varying function b with index $-\alpha^{-1}$, $\alpha > 0$ such that

$$tP(X \in b(t) \cdot) \rightarrow \mu(\cdot), \quad t \rightarrow \infty.$$

The measure μ on the right-hand side must have the property (5.1).

Consider an iid sequence $(X_n)_{n \geq 1}$, where X_n is regularly varying with index $-\alpha < 0$ for each n . Define the point process

$$N_n = \sum_{i=1}^n \delta_{(i/n, X_i/b(n))}. \quad (5.2)$$

It is well-known that regular variation of each X_i is equivalent to the convergence of the point process (5.2) towards a suitable PRM; see Proposition 3.21, [17]. The following theorem is a reproduction of this result under \mathbb{M}_0 -convergence.

Theorem 5.2. Let $(X_n)_{n \geq 1}$ be an iid copies of a random element X taking values in \mathbb{S} and a measure $\mu \in \mathbb{M}_0$. Suppose that there exists an increasing regularly varying function b with index $-\alpha^{-1}$, $\alpha > 0$. A point process N_n is defined as (5.2) and the process N is PRM on $[0, \infty) \times \mathbb{O}$ with mean measure $dt \times d\mu$. Then, $N_n \xrightarrow{w} N$ if and only if

$$tP(X/b(t) \in \cdot) \rightarrow \mu, \quad t \rightarrow \infty, \text{ in } \mathbb{M}_0. \quad (5.3)$$

The proof is simply an application of Laplace functionals. By using Corollary 3.11, it is easy to prove following the line of the proof of Proposition 3.21, [17].

6 Discussions

The choice of the cone \mathbb{C} is vital to the definition of the space \mathbb{M}_0 . This topic is already covered by various literatures in different contexts; see [14, 15, 7, 8], to name a few.

It is proven in [16] that \mathbb{M}_0 -convergence implies vague convergence and $\mathbb{M}_0 \subset \mathbb{M}_+(\mathbb{S} \setminus \mathbb{C})$. The proof is based on the fact that compact sets in \mathcal{S}_0 are bounded away from \mathbb{C} . The application of vague convergence requires that the space \mathbb{S} is locally compact. For this reason, it is tempting to replace vague convergence with the $w^\#$ -convergence in extreme value theory. But the measures in \mathbb{M}_0 does not belong to $\mathbb{M}_\#$, which are the measures frequently used in extreme value theory. One toy example is to take $\mathbb{S} = \mathbb{R}$ and $0_S = 0$. Find a measure $\mu \in \mathbb{M}_0$ such that $\mu(\{x : |x| > 1/r\}) = r$ for all $r > 0$. Note that the set $B_r(0) \setminus \{0\} \in \mathcal{S}_0$ for any fixed r is bounded and $\mu(B_r(0) \setminus \{0\}) > M$ for any $M > 0$. We have $\mu \notin \mathbb{M}_\#(\mathbb{O})$ and hence \mathbb{M}_0 -convergence does not imply $w^\#$ -convergence.

7 Proofs

Proof of Proposition 2.2. The proof is a modification of Proposition A2.5.I and Theorem A2.6.III, [5] under weaker conditions.

Notice that $\mathbb{S} \setminus \mathbb{C}^r$ is close and hence complete and separable. If $\mu \in \mathbb{M}_0$, the measure $\mu^{(r)}$ is in $\mathbb{M}_b(\mathbb{S} \setminus \mathbb{C}^r)$. The space $(\mathbb{M}_b(\mathbb{S} \setminus \mathbb{C}^r), p_r)$ is complete and separable, where p_r is the Prohorov metric. We firstly show that for any $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ and $r > 0$ such that for $\mu \in \mathbb{M}_0$,

$$\{v \in \mathbb{M}_0 : d_{\mathbb{M}_0}(\mu, v) < \varepsilon\} \supset \{v \in \mathbb{M}_0 : p_r(\mu^{(r)}, v^{(r)}) < \delta_1\}, \quad (7.1)$$

$$\{v \in \mathbb{M}_0 : p_r(\mu^{(r)}, v^{(r)}) < \varepsilon\} \supset \{v \in \mathbb{M}_0 : d_{\mathbb{M}_0}(\mu, v) < \delta_2\}. \quad (7.2)$$

For $s > r$ and $\mu, v \in \mathbb{M}_0$, $p_r(\mu^{(r)}, v^{(r)}) \geq p_s(\mu^{(s)}, v^{(s)})$. Let $y = \varepsilon/(2 - \varepsilon)$. Choose $\delta_1 = y/(1 - y)$ and $r = \log(1 + y)$. Then, if $p_r(\mu^{(r)}, v^{(r)}) < \delta_1$

$$d_{\mathbb{M}_0}(\mu, v) \leq \int_0^r e^{-x} dx + \int_r^\infty e^{-x} p_r(\mu^{(r)}, v^{(r)}) dx \leq \varepsilon.$$

Similarly, if $p_r(\mu^{(r)}, v^{(r)}) < \varepsilon$, we can choose $\delta_2 = 1 - e^{-r}(1 + \varepsilon)^{-1}$ and (7.2) holds. According to (7.1) and (7.2), it is enough to consider the open sets in $(\mathbb{M}_b(\mathbb{S} \setminus \mathbb{C}^r), p_r)$ for $r > 0$, which generates \mathcal{M}_0 . In what follows, the connection between the functions Φ_A with $A \in \mathcal{B}(\mathbb{S} \setminus \mathbb{C}^r)$ for some $r > 0$ and the open sets in $(\mathbb{M}_b(\mathbb{S} \setminus \mathbb{C}^r), p_r)$ will be explored.

Let $F \subset \mathbb{S} \setminus \mathbb{C}^r$ be a closed set. Choose $y > 0$ and find $\mu \in \mathbb{M}_0$ such that for some $y > 0$, $\mu(F) < y$ and let $\varepsilon = y - \mu(F)$. Recall that the measures in $\mathbb{M}_b(\mathbb{S} \setminus \mathbb{C}^r)$ are totally finite measures. According to Proposition A2.5.I, [5], the set

$$D = \{v \in \mathbb{M}_0 : \Phi_F(v) < y\} = \{v \in \mathbb{M}_0 : v^{(r)}(F) < \mu^{(r)}(F) + \varepsilon\} \quad (7.3)$$

is an open set, that is, for any $v \in D$, there exists $\delta > 0$ such that

$$\{\tilde{v} : p_r(\tilde{v}^{(r)}, v^{(r)}) < \delta\} \subset D,$$

and by (7.2), there exists $\delta'_2 > 0$ such that

$$\{\tilde{v} : d_{\mathbb{M}_0}(\tilde{v}, v) < \delta'_2\} \subset D.$$

Consequently, D is an open set in \mathbb{M}_0 and hence measurable.

Let \mathcal{A} be the collection of sets $A \in \mathcal{S}_0$ for which Φ_A is \mathcal{M}_0 -measurable. We will show that \mathcal{A} agrees with \mathcal{S}_0 . Trivially, $\mathcal{A} \subset \mathcal{S}_0$. It remain to prove $\mathcal{S}_0 \subset \mathcal{A}$. According to the discussions above, \mathcal{A} contains all the closed set F bounded away from \mathbb{C} , that is, $F \subset \mathbb{S} \setminus \mathbb{C}^r$ for some $r > 0$. Moreover, for any $A, B \in \mathcal{S}_0$ bounded away from \mathbb{C} , the intersection $A \cap B$ is also bounded away from \mathbb{C} . This shows that the sets bounded away from \mathbb{C} forms a π -system. Moreover, \mathcal{C} generates \mathcal{S}_0 . By the definition of Φ , $\Phi_{A \cup B} = \Phi_A + \Phi_B$ if A and B are disjoint and $\Phi_{A \setminus B} = \Phi_A - \Phi_B$ if $B \subset A$. Since

$$A \cap B = (A \cup B) \setminus ((A \cup B) \setminus A) \cup (A \cup B) \setminus B,$$

we have $\Phi_{A \cap B} = \Phi_A + \Phi_B - \Phi_{A \cup B}$. This implies that $\mathcal{D} \subset \mathcal{A}$. For $A \in \mathcal{S}_0$, let $A^{(r_n)} = A \setminus \mathbb{C}^{r_n}$ with $r_n \downarrow 0$ as $n \rightarrow \infty$. The sequence of functions $(\Phi_{A^{(r_n)}})$ is a non-decreasing sequence of measurable functions and for every $\mu \in \mathbb{M}_0$, $\lim_{n \rightarrow \infty} \Phi_{A^{(r_n)}}(\mu) = \Phi_A(\mu)$. Lebesgue's monotone convergence theorem yields that Φ_A is also measurable. By choosing A as \mathbb{O} , we prove that $\Phi_{\mathbb{O}}$ is measurable and consequently, $\mathbb{O} \in \mathcal{A}$. Furthermore, if $A \in \mathcal{A}$, we have $\mathbb{O} \setminus A \in \mathcal{A}$ since $\Phi_{\mathbb{O} \setminus A} = \Phi_{\mathbb{O}} - \Phi_A$ is measurable. Consider a sequence of disjoint set $(A_i)_{i \geq 1}$ with $A_i \in \mathcal{S}_0$. Let $B_n = \bigcup_{i=1}^n A_i^{(r_n)}$ and trivially, the sequence Φ_{B_n}

is non-decreasing with $\lim_{n \rightarrow \infty} \Phi_{B_n} = \Phi_{\cup_i A_i}$ pointwisely. An application of Lebesgue's monotone convergence theorem ensures that $\Phi_{\cup_i A_i}$ is measurable and hence, $\cup_i A_i \in \mathcal{A}$. This shows that \mathcal{A} is a λ -system. According to Dynkin's $\pi - \lambda$ theorem,

$$\mathcal{S}_0 = \sigma(\mathcal{D}) \subset \mathcal{A},$$

which means that $\mathcal{S}_0 = \mathcal{A}$.

It remains to show that \mathcal{M}_0 is the smallest σ -algebra in \mathbb{M}_0 with this property. Let \mathcal{D} be given and let \mathcal{G} be any σ -algebra with respect to which Φ_A is measurable for all $A \in \mathcal{A}$. It follows from the above arguments that Φ_A is \mathcal{G} -measurable for all $A \in \sigma(\mathcal{D}) = \mathcal{S}_0$. Assume that $\mu \in \mathbb{M}_0$, a closed set $F \in \mathcal{S}_0$ bounded away from \mathbb{C} and $y > 0$ are given. The set D defined in (7.3) is then a open set of \mathbb{M}_0 and an element of \mathcal{G} , which implies that \mathcal{G} contains a basis for the open sets of \mathcal{M}_0 . Since $(\mathbb{M}_0, d_{\mathbb{M}_0})$ is separable, any open set of \mathcal{M}_0 can be written as a countable union of basic sets and thus, all open sets are in \mathcal{G} . Therefore, \mathcal{G} contains \mathcal{M}_0 , which completes the proof. \square

Proof of Lemma 3.9. According to Theorem A2.4.I, [5] and Theorem 2.5, [16], the set $K \in \mathcal{M}_0$ is compact if there exists a sequence (r_i) with $r_i \downarrow 0$ such that for each i and any $\varepsilon > 0$, there exist constant $M_i > 0$ and compact sets $K_{i,\varepsilon'} \subset \mathbb{S} \setminus \mathbb{C}^{r_i}$ such that

$$\sup_{\xi \in K} \xi(\mathbb{S} \setminus \mathbb{C}^{r_i}) < M_i, \quad (7.4)$$

$$\sup_{\xi \in K} \xi(\mathbb{S} \setminus (K_{i,\varepsilon'} \cup \mathbb{C}^{r_i})) \leq \varepsilon. \quad (7.5)$$

Suppose that (3.4) and (3.5) are satisfied. From (3.4), we choose \widetilde{M}_i such that

$$P_t(\xi(\mathbb{S} \setminus \mathbb{C}^{r_i}) > \widetilde{M}_i) < \varepsilon/2^{n+1},$$

and from (3.5), we choose the compact set $\widetilde{K}_{ij} \subset \mathbb{S} \setminus \mathbb{C}^{r_i}$ such that

$$P_t(\xi(\mathbb{S} \setminus (\widetilde{K}_{ij} \cup \mathbb{C}^{r_i})) > j^{-1}) < \varepsilon/2^{m+n+2}.$$

Define the sets, for $i, j \geq 1$,

$$Q_i = \{\xi : \xi(\mathbb{S} \setminus \mathbb{C}^{r_i}) \leq \widetilde{M}_i\},$$

$$Q_{i,j} = \{\xi : \xi(\mathbb{S} \setminus (\widetilde{K}_{ij} \cup \mathbb{C}^{r_i})) \leq j^{-1}\}.$$

Let $K = \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{\infty} (Q_i \cap Q_{i,j})$ and (7.4) and (7.5) are satisfied by construction. Hence, K is compact and

$$P_t(K^c) \leq \sum_{i=1}^{\infty} \left[P_t(Q_i^c) + \sum_{j=1}^{\infty} P_t(Q_{i,j}^c) \right] \leq \sum_{i=1}^{\infty} \left[\frac{\varepsilon}{2^{n+1}} + \sum_{j=1}^{\infty} \frac{\varepsilon}{2^{m+n+2}} \right] = \varepsilon.$$

Thus, $(P_t)_{t \in \mathcal{T}}$ are uniformly tight.

Suppose that the measures $(P_t)_{t \in \mathcal{T}}$ are uniformly tight. For a given $\varepsilon > 0$, a compact set $K \in \mathcal{M}_0$ exists and consequently, there exists a sequence (r_i) with $r_i \downarrow 0$ such that for each i and any $\varepsilon > 0$, there exist constants $M_i > 0$ and compact sets $K_{i,\varepsilon'} \subset \mathbb{S} \setminus \mathbb{C}^{r_i}$ such that for all $\xi \in K$, (7.4) and (7.5) hold. Consequently, (3.4) and (3.5) are satisfied. \square

Proof of Theorem 3.10. Lemma 3.8 have proved the first part of the theorem. Now we need to prove the other part. Since a probability measure is uniquely determined by the class of all fidi distributions, it suffices to show that the family (P_n) is uniformly tight, which implies that if the fidi distributions of a subsequence of (P_n) converges weakly to

those of P , then all convergent subsequence have the same limit and the whole sequence (P_n) converges weakly to P .

We then need to check that (3.4) and (3.5) are satisfied by using the assumption of the convergence of fidi distributions. By Lemma 3.7, a sequence $(r_i)_{i \geq 1}$ exists such that $r_i \downarrow 0$ and for all i , \mathbb{C}^{r_i} are stochastic continuity sets for all P_n , $n \geq 1$ and P . For a given i and $\varepsilon > 0$, we can find a constant $M > 0$ that is continuity point for the distribution $\xi(\mathbb{S} \setminus \mathbb{C}^{r_i})$ and for which $P(\xi(\mathbb{S} \setminus \mathbb{C}^{r_i}) > M) \leq \varepsilon/2$ and $P_n(\xi(\mathbb{S} \setminus \mathbb{C}^{r_i}) > M) \rightarrow P(\xi(\mathbb{S} \setminus \mathbb{C}^{r_i}))$. For some $N > 0$ and $n > N$, $P_n(\xi(\mathbb{S} \setminus \mathbb{C}^{r_i}) > M) < \varepsilon$ and then we can ensure that (3.4) holds for all n by increasing M .

Since (\mathbb{S}, d) is complete and separable and the sets $\mathbb{S} \setminus \mathbb{C}^{r_i}$, $i \geq 1$, are closed, the subspaces $(\mathbb{S} \setminus \mathbb{C}^{r_i}, d)$ are also complete and separable. For each i , we can find a countable dense set $D_i = \{x_j, j \geq 1\}$ of $\mathbb{S} \setminus \mathbb{C}^{r_i}$. By Lemma 3.7, for any $j, l \geq 1$, we can find a neighborhood of x_j , $B_{jl} = B_{r_l}(x_j)$ such that $r_j \leq \min\{d(x_j, 0), 2^{-l}\}$ and B_{jl} is a stochastic continuity set for P . Trivially,

$$\xi\left(\bigcup_{j=1}^K B_{jl}\right) \uparrow \xi(\mathbb{S} \setminus \mathbb{C}^{r_i}).$$

Given $\varepsilon, \varepsilon' > 0$, we can choose K_l such that

$$P\left(\xi(\mathbb{S} \setminus (\mathbb{C}^{r_i} \cup B_l)) \geq \varepsilon'_j\right) \leq \varepsilon/2^{j+1},$$

where $B_l = \bigcup_{j=1}^{K_l} B_{jl}$ and $\varepsilon'_j \leq \varepsilon'/2^j$ is chosen to be a continuity point of the distribution of $\xi(\mathbb{S} \setminus (\mathbb{C}^{r_i} \cup B_l))$ under P . Using the weak convergence of the fidi distribution and increasing the value of K_l if necessary, we can ensure that for all n ,

$$P_n\left(\xi(\mathbb{S} \setminus (\mathbb{C}^{r_i} \cup B_l)) \geq \varepsilon'_j\right) \leq \varepsilon/2^j.$$

Define $K_i = \bigcap_{l=1}^{\infty} B_l$. By construction, K_i is closed and it can be covered by a finite number of ε -spheres for every $\varepsilon > 0$. Consequently, K_i is compact. Moreover, for every n ,

$$\begin{aligned} P_n\left(\xi(\mathbb{S} \setminus \mathbb{C}^{r_i}) - \xi(K_i) > \varepsilon'\right) &= P_n\left(\xi\left(\bigcup_{l=1}^{\infty} (\mathbb{S} \setminus (\mathbb{C}^{r_i} \cup B_l))\right) > \varepsilon'\right) \\ &\leq \sum_{l=1}^{\infty} P_n\left(\xi(\mathbb{S} \setminus (\mathbb{C}^{r_i} \cup B_l)) > \varepsilon'/2^j\right) \\ &\leq \sum_{l=1}^{\infty} P_n\left(\xi(\mathbb{S} \setminus (\mathbb{C}^{r_i} \cup B_l)) > \varepsilon'_j\right) \\ &\leq \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon, \end{aligned}$$

and thus (3.5) holds. This completes the proof. \square

Proof of Lemma 4.4. Let

$$\mathcal{A} = \{f : f \in BM(\widetilde{\mathbb{S}}), f \text{ satisfies (4.6)}\}.$$

Think of the functions of the form $f(y, x) = f_1(y)f_2(x) \in [0, 1]$ where f_1 is non-negative

and \mathcal{K} -measurable and $f_2 \in BM(\mathbb{S})$. An application of (4.4) yields that

$$\begin{aligned} E\left(f(K_i, X_i) \mid (X_n)_{n \geq 1}\right) &= E\left(f_1(K_i) f_1(X_i) \mid (X_n)_{n \geq 1}\right) \\ &= f_2(X_i) E\left(f_1(K_i) \mid (X_n)_{n \geq 1}\right) \\ &= f_2(X_i) \int_{\mathbb{K}} f_1(y) G(dy, X_i) \\ &= \int_{\mathbb{K}} g(y, X_i) G(dy, X_i). \end{aligned}$$

This implies that the indicator function $\mathbf{1}_{F_i \times A_i} = \mathbf{1}_{F_i} \mathbf{1}_{A_i}$ belongs to \mathcal{A} , where $A_i \in \mathcal{S}_0$ bounded away from \mathbb{C} and $F_i \in \mathcal{K}$.

Fix $r > 0$. Let $BM(\tilde{\mathbb{S}} \setminus \tilde{\mathbb{C}}^r)$ be a collection of functions $g \in \mathcal{A}$ such that g vanishes on $\tilde{\mathbb{C}}^r$ and let $\mathcal{E}^r = \{G \in \tilde{\mathcal{S}}_0 : \mathbf{1}_G \in BM(\tilde{\mathbb{S}} \setminus \tilde{\mathbb{C}}^r)\}$. Following the arguments in the proof of Proposition 2.2, it is trivial to show that \mathcal{E}^r is a λ -system containing the π -system of rectangles $F \times A$ with $A \in \mathcal{B}(\mathbb{S} \setminus \mathbb{C}^r)$ and $F \in \mathcal{K}$. Dynkin's theorem shows that $\mathcal{E}^r \supset \mathcal{B}(\tilde{\mathbb{S}} \setminus \tilde{\mathbb{C}}^r)$. Moreover, the simple function

$$\sum_{i=1}^k c_i \mathbf{1}_{\tilde{F}_i} \in BM(\tilde{\mathbb{S}} \setminus \tilde{\mathbb{C}}^r), \quad (7.6)$$

where $c_i \in [0, 1]$ and $\tilde{F}_i \in \mathcal{B}(\mathbb{S} \setminus \mathbb{C}^r) \times \mathcal{K}$, $i = 1, \dots, k$. Any measurable function $f : \tilde{\mathbb{S}} \setminus \tilde{\mathbb{C}}^r \rightarrow [0, 1]$ in $BM(\tilde{\mathbb{S}} \setminus \tilde{\mathbb{C}}^r)$ can be written as the monotone limit of simple functions of the form (7.6) and Lebesgue's monotone convergence theorem yields that $f \in \mathcal{A}$. Since r is arbitrary, if $g \in BM(\tilde{\mathbb{S}})$, there exists $r > 0$ such that g vanishes on $\tilde{\mathbb{C}}^r$ and consequently, $g \in \mathcal{A}$.

For $g \in \mathcal{A}$ of the form $g(y, x) = g_1(y)g_2(x)$, it is easy to verify that (4.7) is satisfied. Using similar arguments as above, it is trivial to show that (4.7) holds. \square

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